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General Class of Estimators

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SUMMARY

To estimate any parameter of a finite population, using auxiliary information, a general class of estimators is proposed. Its minimum mean square error is obtained to the first degree of approximation. Many classes of estimators proposed in the literature are identified as its particular cases.
Key Words : Asymptotic mean squre error, Auxiliary information, Taylor's series.

Introduction and Notations

The problem of estimating the parameter F_0 of a study variable, y, is considered using supplementary information on an auxiliary variable. Let $F_1, F_2, ... F_p$ be the parameters of the auxiliary character x which are assumed to be known. Let $f_0, f_1, f_2, ..., f_p$ be the unbiased or consistent estimators of $F_0, F_1, .., F_p$, respectively, each based on a sample of size $n > p$. Let $\bar{u} = (u_1, u_2, ..., u_p)$, where, $u_1 = f_i/F_i$, $i = 1, 2, \ldots, p$ assume values in a bounded closed, convex subset R_p of p-dimensional real space containing the point $\bar{\varepsilon} = (1, 1, ..., 1)$.

$$\text{Let} \quad \bar{b}^t = (b_1, b_2, ..., b_p), \text{ where } b_i = \frac{F_0 \operatorname{Cov}(f_0, f_i)}{F_i V(f_0)}, i = 1, 2, 3, ..., p$$

$$\text{and} \quad A = [a_{ij}]_{p \times p}, \text{ where } a_{ij} = \frac{F_0^2 \operatorname{Cov}(f_i, f_j)}{F_i F_j V(f_0)}; i, j = 1, 2, 3, .., p.$$

The matrix A is assumed to be positive definite.

Writing

$$e_0 = (f_0/F_0) - 1, \; \varepsilon' = (e_1, e_2, ... e_p)$$

where

$$e_i = u_i - 1, C_0^2 = V(f_0)/F_0^2, C_i^2 = V(f_i)/F_i^2$$

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$$C_{0i} = \text{Cov}\,(f_0, f_i)/(F_0 F_i) \text{ and } C_{ij} = \text{CoV}\,(f_i, f_j)/(F_i F_j)$$

We have $E(e_i) = B(f_i)/F_i,\ i = 0, 1, 2, \ldots, p$

where $B(f_i)$ denotes the bias in the estimator f_i of F_i and

$$E(e_0^2) = C_0^2,\ E(e_i^2) = C_i^2,\ E(e_0 e_i) = C_{0i},$$

$$E(e_0\,\varepsilon) = C_0^2 b,\ E(\varepsilon\varepsilon') = C_0^2 A,\ i = 1, 2, \ldots, p.$$

2. *Proposed Class*

One may propose a general class of estimators as

$$t_h = h\,[f_0, \overline{u}] \tag{2.1}$$

where $h\,[f_0, u]$ be a function of f_0 and u_i, $i = 1, 2, \ldots, p$ such that

$$h\,[F_0, \overline{\varepsilon}] = F_0 \tag{2.2}$$

and h is bounded and continuous with bounded and continuous first and second order partial derivatives in R_{p+1}.

Expanding $h\,[f_0, \overline{u}]$ about the point $(F_0, \overline{\varepsilon})$ in a second order Taylor's series, we get

$$t_h = h(F_0, \overline{\varepsilon}) + (f_0 - F_0) h_0'(F_0 \overline{\varepsilon}) + \sum_{i=1}^{p} (\mu_i - 1)\, h_i'(F_0, \overline{\varepsilon})$$

$$+ \sum_{i<j}^{p} (\mu_i - 1)(\mu_j - 1)\, h_{ij}''(F_0, \overline{\varepsilon}) + \sum_{i=1}^{D} (\mu_i - 1)^2 h_{ii}''(F_0, \overline{\varepsilon})$$

$$+ \sum_{i=j}^{p} (\mu_i - 1)(f_0 - F_0)\, h_{0i}''(F_0, \overline{\varepsilon}) + (f_0 - F_0)^2 h_{00}''(F_0, \overline{\varepsilon}) + \cdots \tag{2.3}$$

Using (2.2) and further assuming that $h_0'(F_0, \overline{\varepsilon}) = 1$ by following Srivastava [5] and Srivastava and Jhajj ([7], [8], [9], [10]) , we get

$$t_h = f_0 + \sum_{i=1}^{p} (\mu_i - 1)\, h_i'(F_0, \overline{\in}) + \sum_{i<j}^{p} (\mu_i - 1)(\mu_j - 1)\, h_{ij}''(F_0, \overline{\in})$$

$$+ \sum_{i=1}^{p} (\mu_i - 1)^2 h_{ii}''(F_0, \overline{\in}) + \sum_{i=1}^{p} (\mu_i - 1)(f_0 - F_0)\, h_{0i}''(F_0, \overline{\in})$$

$$+ (f_0 - F_0)^2 h_{00}''(F_0, \overline{\in}) + \cdots \tag{2.4}$$

where $h_i'(F_0, \overline{\in})$ and $h''_{ij}(F_0, \overline{\in})$, $i, j = 0, 1, 2, .., p$ denote the first and second order partial derivatives of $h(f_0, \overline{u})$ with respect to f_0 and u_i, respectively.

Taking expected value on both sides of (2.4), we get

$$E(t_h) = F_0 + \sum_{i<j}^{p} C_{ij} h_{ij}''(F_0, \overline{\varepsilon}) + \sum_{i=1}^{p} C_{ii} h_{ii}''(F_0, \overline{\varepsilon}) + F_0^2 C_{00} h_{00}''(F_0, \overline{\varepsilon})$$

$$+ F_0 \sum_{i=1}^{p} C_{0i} h_{0i}''(F_0, \overline{\varepsilon}) + B(f_0) + \sum_{i=1}^{p} B(f_i) h_i'(F_0, \overline{\varepsilon})/F_i \qquad (2.5)$$

Since $f_0, f_1, f_2, ..., f_p$ are either unbiased or consistent estimators of $F_0, F_1, F_2, ... F_p$, respectively. Therefore by following the definition of consistency from Gujarati [2], $C_{ij}(i, j = 0, 1, 2, .., p)$ representing the variance-covariance terms and $B(f_i)$ will tend to zero as $n \longrightarrow \infty$, so that (2.5) can be written as

$$E(t_h) = F_0 + O(n^{-1}) \qquad (2.6)$$

that is, the bias in the class of estimators is of order, n^{-1} and mean square error (MSE) of the class of estimators t_h is given by

$$MSE(t_h) = E(t_h - F_0)^2$$

$$= E\left[f_0 + \sum_{i=1}^{p} (\mu_i - 1) h_i(F_0, \overline{\varepsilon}) - F_0\right]^2$$

$$= F_0^2 C_0^2 + \{h_m'(F_0, \overline{\varepsilon})\}' C_0^2 A h_m'(F_0, \overline{\varepsilon})$$

$$+ 2C_0^2 b' h_m'(F_0, \overline{\varepsilon}) h_0(F_0, \overline{\varepsilon}) F_0 \qquad (2.7)$$

where $h_m(F_0, \overline{\varepsilon}) = \{h_i(F_0, \overline{\varepsilon})\}_{px1}$. The optimum values of the parameters in $h(. , . . .)$ to minimize the MSE of t_h at (2.7) are given by

$$h_m(F_0, \overline{\varepsilon}) = F_0 A^{-1} b \qquad (2.8)$$

The resultant (minimum) MSE of t_h is given by

$$\text{Min. } M(t_h) = V(f_0)(1 - \overline{b}' A^{-1} \overline{b}) = V(f_0)(1 - R^2) \qquad (2.9)$$

where R stands for the multiple correlation coefficient between f_0 and $(f_1, f_2, ... f_p)$. Since multiple correlation coefficient increases with increasing number of secondary variables it follows from (2.9) that the minimum mean square error of t_h is monotone decreasing function of p.

3. *Particular cases*

Various classes of estimators of population mean, population variance, population correlation coefficient etc. may be identified as particular cases of the class of estimators t_h. For example, with suitable chosen parameters in h (.,.), the following classes by

(1) Srivastava [6] : $g(\hat{Y}, \hat{X})$; for $p = 1$, $h(F_0, 1) = F_0$, $f_0 = \hat{Y}$

(2) Srivastava and Jhajj [9] : $g[\bar{y}, \bar{x}/\bar{X}, s_x^2/S_x^2]$ for $p = 2$, $U_1 = \bar{x}/\bar{X}$,

$u_2 = s_x^2/S_x^2$, $h(F_0, 1, 1) = F_0$, $f_0 = \bar{y}$

(3) Srivastava and Jhajj [8] : $g(s_y^2, \bar{x}/\bar{X}, s_x^2/S_x^2)$ for $p = 2$,

$u_1 = \bar{x}/\bar{X}$, $u_2 = s_x^2/S_x^2$, $h(F_0, 1, 1) = F_0$, $f_0 = s_y^2$

(4) Srivastava and Jhajj [11] : $h(r, \bar{x}/\bar{x}, s_x^2/S_x^2)$ for $p = 2$,

$u_1 = \bar{x}/\bar{X}$, $u_2 = s_x^2/S_x^2$, $h(F_0, 1, 1) = F_0$, $f_0 = r$

(5) Srivastava and Jhajj [10] :

(i) $h(\bar{y}, \bar{x}/\bar{X}, s_x^2/S_x^2, r/\rho)$ for $p = 3$, $u_i = \bar{x}/\bar{X}$,

$u_2 = s_x^2/S_x^2$, $u_3 = r/\rho$, $h(F_0, 1, 1, 1) = F_0$, $f_0 = \bar{y}$

(ii) $h(s_y, \bar{x}/\bar{X}, s_x^2/S_x^2, r/\rho)$ for $p = 3$, $u_i = \bar{x}/\bar{X}$,

$u_2 = s_x^2/S_x^2$, $u_3 = r/\rho$, $h(F_0, 1, 1, 1) = F_0$, $f_0 = s_y^2$

are the special cases of the proposed class.

(6) Since $\hat{\beta} = s_{xy}/S_x^2$ is an unbiased estimator of regression coefficient β, so one may say that if β is known then

(i) $h(\bar{y}, \bar{x}/\bar{X}, s_x^2/S_x^2, \hat{\beta}/\beta)$ for $p = 3$, $u_1 = \bar{x}/\bar{X}$, $u_2 = s_x^2/S_x^2$
$u_3 = \hat{\beta}/\beta$, $h(F_0, 1, 1, 1) = F_0$, $f_0 = \bar{y}$

(ii) $h(s_y^2, \bar{x}/\bar{X}, s_x^2/S_x^2\, \hat{\beta}/\beta)$ for $p = 3$, $u_1 = \bar{x}/\bar{X}$, $u_2 = s_x^2/S_x^2$
$u_3 = \hat{\beta}/\beta$, $h(F_0, 1, 1, 1) = F_0$, $f_0 = s_y^2$

are also the special cases of the proposed class.

Remark : The classes of estimators proposed by Biradar and Singh [1], Singh and Kataria [3], Singh [5] and Singh and Upadhyaya [4] are not shown to be the special cases of the propsed general class of estimators as Srivastava [7] has shown that these classes of estimators are not improvements over the original one.

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Sampling Without Replacement in Qualitative Randomized Response Model

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Summary

It is shown that the probability of 'yes' response is same for simple random sampling with replacement (SRSWR) as well as without replacement (SRSWOR). Some estimators have been derived for SRSWOR and their variances obtained using randomized response models for binary and discrete data. Estimators developed under SRSWOR are more efficient than estimators under SRSWR irrespective of randomized response model used, provided N is finite. The mean square error of estimators of randomized response model under SRSWOR and SRSWR are compared with the MSE of conventional estimator under various assumptions about the underlying population. This study established the supremacy of unrelated question randomized response model under SRSWOR over open interview with nominal untruthful reporting of order 5% under the same scheme.

Key words : Sensitive attribute; Randomised response; Unbiased estimate.

Introduction

Warner [4] developed a model for estimating the proportion of individuals possessing a sensitive attribute without requiring the individual respondent to report to the interviewer whether or not he possesses the sensitive attribute. The technique consists in presenting a random device to the respondent, (say) a spinner with a face mark such that the spinner points to the letter A with probability P and to letter $\bar{A}$ with probability $(1 - P)$, $P \neq \frac{1}{2}$ where A represents the sensitive attribute and $\bar{A}$ the non-sensitive attribute (complement of A). A simple random sample of n individuals is drawn with replacement. The respondent is asked to spin the spinner unobserved by the interviewer and report only 'yes' or 'No', whether or not the spinner points to the letter representing the group to which the interviewer belongs. The response to either question will divide the sample space into two mutually exclusive and complementary classes.

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